*Please reply to problem 1 and to problems 2,3,4,5 on separate sheets of paper.*

# 1 Pendulum on a cart

We consider a pendulum in the gravity field of the earth. The pendulum is fixed on a cart (see Figure). The cart (of mass M) slides frictionless on a horizontal plane. The pendulum consists of a mass m, attached to a massless string of length l that is attached to a stick at point A. The stick is fixed rigidly to the cart.
The string is initially kept in a horizontal position, that is, the mass m is at the same height as point A. At time $t = 0$, the mass m is released from this height with velocity zero.

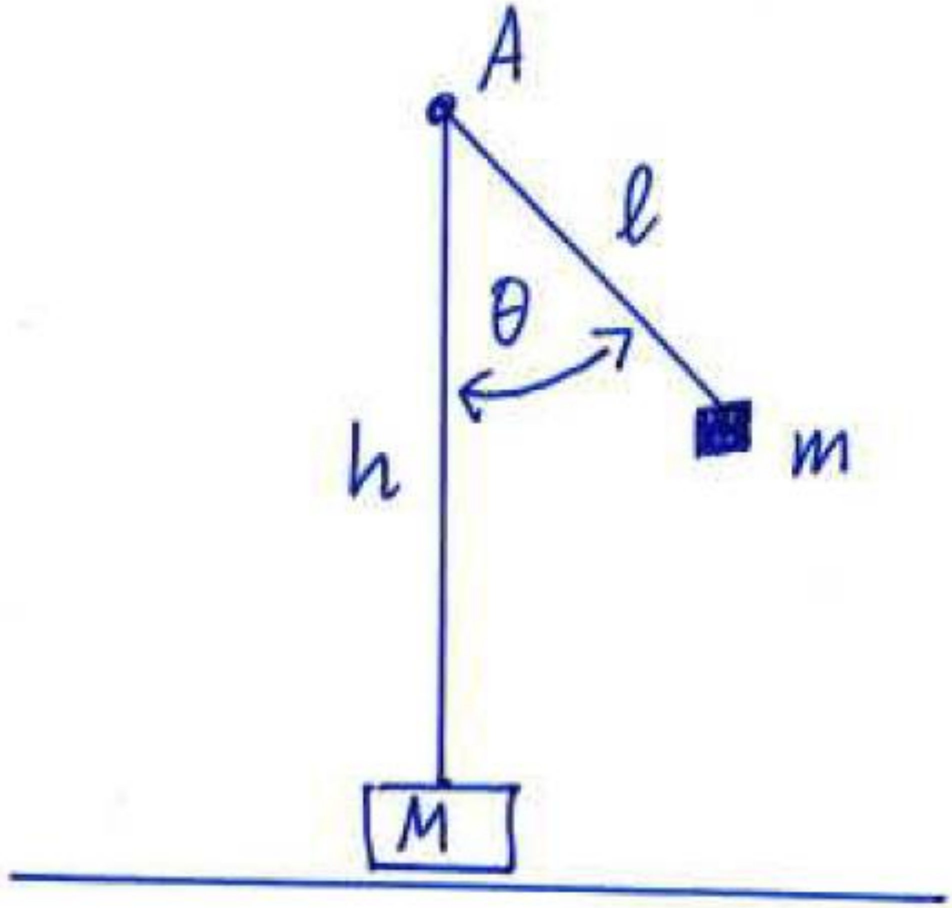


Figure 1: Pendulum on a cart.

*NB.: Several questions below ask for* **qualitative** *assessments. This means that your answer should be concise but argued in full sentences in a logical manner. Make sure you give the physical reasoning for any claims you make. You can give mathematical expressions for any quantities that you consider useful to discuss, but equations should not replace an argumented answer.*

1. In this question, we consider the case M >> m. Describe the motion of the different parts of the system first qualitatively, then quantitatively.
2. In the general case of given M and m, describe *qualitatively* the motion of the different parts of the system.
3. Which physical quantities are conserved quantities? Why? Give their values in terms of the constants of the problem.
4. Explain why the state of the system at a time $t$ after releasing the mass m can be fully characterized by an angle $\theta$ that gives the deviation of the string from the vertical line. Express in particular the position of the cart in terms of $\theta$.
5. When $\theta$=0, that is, when the mass reaches its minimum height and the string is in vertical position, what are the kinetic energies of m and M? What is the velocity of m at this point? What about M?
6. Derive a differential equation for $\frac{d\theta}{dt}$.
7. Give formally the solution of the above differential equation, in the form of an integral. *Do not attempt to integrate!*
8. In the following, we consider that m is a box of sand. Unfortunately, the box has a hole so that sand spills out of the box at speed $\frac{dm}{dt} = c$. We go back to the case M>>m. How does the description of question 1 change?
9. In the general case, describe qualitatively how the motion of the different parts of the system is changed when the sand spills out.
10. Describe qualitatively the trajectory of the sand.
11. We now return to the situation where the mass m stays constant over the time of the experiment. However, we assume in this question that m and M carry electrical charges (denoted respectively $Q_m$ and $Q_M$). Discuss if the fact that the masses are charged can *qualitatively* change the motion of m and M for the case $Q_m = Q_M = q$.

# 2 Computing electric fields and potentials using Gauss's law

A very long cylindrical object consists of an inner solid cylinder of radius $a$, which has a uniform charge density $\rho$, and a concentric thin cylinder, of radius $b$, which has an equal but opposite total charge, uniformly distributed on the surface, see figure 2.

1. Using the symmetries of the problem, calculate the electric field everywhere.
2. Calculate the electric potential everywhere, taking $V = 0$ on the outer cylinder.

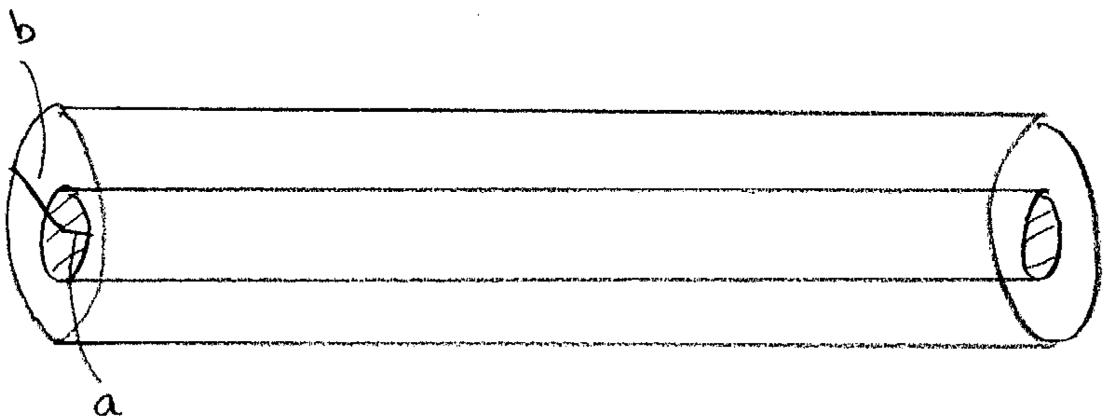


Figure 2: Cylindrical object.

3. Calculate the electrostatic energy $W$ per unit length of the object, using either one of the following formulae for $W$,

$$W = \frac{1}{2}\epsilon_0 \int \mathbf{E} \cdot \mathbf{E}\, d^3x = \frac{1}{2} \int \rho V d^3x \,. \tag{2.1}$$

# 3 Magnetic dipoles

To date, no magnetic monopoles have been discovered in nature. The next simplest field configuration arises from a magnetic dipole, which can be modelled by an infinitesimal loop carrying current. Unlike a point charge or monopole, a dipole $\mathbf{m}$ has both a magnitude and a direction. For a plane loop of area $A$ carrying current $I$, the magnetic moment is given by $\mathbf{m} = AI\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ the unit normal vector to the area with direction determined by the current in accord with the right hand rule.

The force on a magnetic dipole $\mathbf{m}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \,. \tag{3.1}$$

1. Derive (3.1) for a dipole generated by a loop that is an infinitesimal square with side length $\epsilon$. For simplicity, assume that the loop lies in the $yz$-plane with corners at $(0, 0, 0)$, $(0, \epsilon, 0)$,$(0, 0, \epsilon)$, $(0, \epsilon, \epsilon)$. Take the current to flow from $(0, 0, 0)$ to $(0, \epsilon, 0)$ and so forth around the loop. Calculate the Lorentz force on each of the four sides by expressing $\mathbf{B}$ in terms of its Taylor series around the origin to first order. Show that this reproduces (3.1) for the case at hand.

2. We now want to show that the force that a magnetic dipole $\mathbf{m_1}$ exerts on a dipole $\mathbf{m_2}$, when $\mathbf{m_1}$ and $\mathbf{m_2}$ lie on the $z$-axis a distance $r$ apart and are oriented in the positive

$z$ direction, is given by

$$\mathbf{F} = -\frac{3\mu_0 m_1 m_2}{2\pi r^4}\hat{\mathbf{z}}\,. \tag{3.2}$$

Recall that the magnetic field $\mathbf{B}$ at position $\mathbf{r}$ generated by a magnetic dipole $\mathbf{m}$ situated at the origin is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m})\,. \tag{3.3}$$

(a) Express the equation for the $\mathbf{B}$-field (3.3) in spherical coordinates.

(b) Compute the force that $\mathbf{m_1}$ exerts on $\mathbf{m_2}$. Recall that the gradient in spherical coordinates of a function $f$ is given by $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$.

(c) How does this force change when the dipoles are oriented in opposite directions?

# 4 The Meissner effect

The electric field $\mathbf{E}$ vanishes inside a perfect conductor, and any net charge resides on the surface.

1. Show that the magnetic field is constant, i.e. $\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$, inside a perfect conductor.
2. Show that the magnetic flux through a perfectly conducting loop is constant.

A superconductor, in addition to being a perfect conductor, also expels magnetic fields, i.e $\mathbf{B} = \mathbf{0}$ inside a superconductor. This property is called the Meissner effect.

3. Show that the current in a superconductor is confined to the surface.

Charges moving on a two-dimensional surface give rise to a surface current $\mathbf{K}$. Given a surface charge density $\sigma$ and an associated velocity field $\mathbf{v}$, $\mathbf{K} = \sigma\mathbf{v}$.

Assume that a superconductor occupies the entire half-space below the $xy$-plane, such that the $xy$-plane describes the surface of the superconductor.

4. Show that the normal component of a $\mathbf{B}$-field is continuous across the $xy$-plane.
5. Determine the discontinuity in the tangential component of the $\mathbf{B}$-field in terms of the surface current $\mathbf{K}$.
6. Argue that the results from the two previous subquestions can be summarized in the equation $\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$, where $\hat{\mathbf{n}}$ is a unit normal vector to the surface pointing upward (i.e. in the positive $z$-direction).

A familiar demonstration of superconductivity is the levitation of a magnet over a piece of superconducting material. We will analyze this phenomenon here using the method of images. Treat the magnet as a perfect dipole $\mathbf{m}$, a height $h$ above the origin, and constrained to point in the $z$ direction, i.e. $\mathbf{m} = m\hat{\mathbf{z}}$. Recall that because of the Meissner effect, $\mathbf{B} = \mathbf{0}$ for $z \leq 0$.

7. What is the value of $B_z$ right above the surface of the superconductor?
8. Replace the superconductor by an image dipole to reproduce this value of $B_z$. Give magnitude, orientation, and position of the dipole.
9. At what height $h$ over the surface does the magnet, of mass $M$ and subject to the earth's gravitational field, levitate? You can use the result from question 2.2.

The magnetic field we attributed to the image dipole is in fact due to a surface current $\mathbf{K}$ induced by the presence of the magnet. This current can be determined by considering the tangential component of the $\mathbf{B}$-field in the configuration described. The magnetic field due to a magnetic dipole is given in equation (3.3).

10. Using the relation determined above between the tangential component of the $\mathbf{B}$-field and the surface current $\mathbf{K}$, show that the surface current is given by

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2 + h^2)^{5/2}}\hat{\boldsymbol{\phi}}\,,$$

with $r$ the distance from the origin.

# 5 Force on a charge in the presence of conducting planes

Suppose that the entire $xz$ and $yz$ planes are conducting. Calculate the force $\mathbf{F}$ on a particle of charge $q$ located at $x = x_0$, $y = y_0$, $z = 0$ by using the method of images and replacing the planes by image charges.